inkMaths

ANSWER BOOK

Arc length and surface area

Further Pure 2 · Year FM

6 questions · 22 marks

NAVY EDITION

SECTION A · Warm-up

A1 $dy/dx = x^{1/2}$, so $1 + (dy/dx)^2 = 1 + x$ and the integrand is $\sqrt{(1 + x)}$. Integrating gives $(2/3)(1 + x)^{3/2}$, which runs from 2/3 to 16/3. The length is 14/3, about 4.667. [3]

A2 $dy/dx = \sinh x$ and $1 + \sinh^2 x = \cosh^2 x$, so the integrand is $\cosh x$ itself: the one curve for which the square root disappears. The length is $[\sinh x]$ from 0 to 1 = sinh 1, about 1.175. [3]

SECTION B · Standard

B1 $dx/dt = 2t$ and $dy/dt = 3t^2$, so the integrand is $\sqrt{(4t^2 + 9t^4)} = t\sqrt{(4 + 9t^2)}$ for $t \geq 0$. Substituting $u = 4 + 9t^2$ gives $(1/27)(4 + 9t^2)^{3/2}$, which runs from 8/27 to $13^{3/2}/27$. The length is $(13\sqrt{13} - 8)/27$, about 1.440. [4]

B2 In polar form the integrand is $\sqrt{(r^2 + (dr/d\theta)^2)}$. Here $dr/d\theta = e^\theta$ as well, so the bracket is $2e^{2\theta}$ and the integrand is $\sqrt{2}\, e^\theta$. Integrating gives $\sqrt{2}(e^\pi - 1)$, about 31.31. Every polar arc length needs r and its derivative, never r alone. [4]

B3 The surface integral is $2\pi\int y\, ds = 2\pi\int x^3\sqrt{(1 + 9x^4)}\, dx$ from 0 to 1. Substituting $u = 1 + 9x^4$ gives $(\pi/27)(1 + 9x^4)^{3/2}$, which runs from $\pi/27$ to $10\sqrt{10}\, \pi/27$. The area is $(\pi/27)(10\sqrt{10} - 1)$, about 3.563 square units. [4]

SECTION C · Stretch

C1 Since $ds = \cosh x\, dx$, the integral is $2\pi\int \cosh^2 x\, dx$ from 0 to 1. Writing $\cosh^2 x = (\cosh 2x + 1)/2$ gives $2\pi[x/2 + \sinh(2x)/4]$, so the area is $2\pi(1/2 + (\sinh 2)/4) = \pi + \pi(\sinh 2)/2$, about 8.839 square units. The identity is what makes the integral elementary. [4]