inkMaths

ANSWER BOOK

Areas, parametric curves and the limit of a sum

Integration · Year 12–13

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 Crossings where $x = x^2$: $x = 0$ and 1. The line rides above the parabola between them, so the area is $\int_0^1 (x - x^2)\,dx = 1/2 - 1/3 = 1/6$. Top minus bottom, integrated across the enclosed stretch. [3]

SECTION B · Standard

B1 Crossings: $9 - x^2 = x^2 + 1$ gives $x^2 = 4$, so $x = \pm 2$. The gap is $(9 - x^2) - (x^2 + 1) = 8 - 2x^2$, and $\int_{-2}^{2} (8 - 2x^2)\,dx = [8x - 2x^3/3]$ from -2 to $2 = 64/3$. One integral of the height difference handles both boundaries at once. [4]

B2 $dx/dt = 3t^2$, so the area is $\int_0^2 3t^2 \times 3t^2\,dt = \int_0^2 9t^4\,dt = (9/5)(32) = 288/5$. Cartesian check: $t = x^{1/3}$ gives $y = 3x^{2/3}$, and $\int_0^8 3x^{2/3}\,dx = (9/5)x^{5/3}$ at $8 = 288/5$. Staying in the parameter is the shorter road; the check is optional insurance. [4]

B3 The limit is $\int_1^3 x^2\,dx = [x^3/3]$ from 1 to 3 = 26/3. Recognition is the whole task: $f(x)\,\delta x$ is a strip's area, Σ adds the strips, and the limit turns the sum into the stretched S. [3]

SECTION C · Stretch

C1 $dx/dt = -3 \sin t$, and as t runs 0 to $\pi/2$, x runs backwards from 3 to 0; the area is $\int y\,(dx/dt)\,dt$ with the orientation absorbed: $3\int_0^{\pi/2} \sin^2 t\,dt$. Rewriting $\sin^2 t = \tfrac{1}{2} - \tfrac{1}{2}\cos 2t$ gives $3[t/2 - (\sin 2t)/4]$ from 0 to $\pi/2 = 3\pi/4$. The identity from the standard-functions lesson is the key that unlocks the parametric area. [4]

C2 The definite integral is defined as the limit of a sum of strip areas $\Sigma f(x)\,\delta x$ as the strips shrink. The S records that ancestry: integration is summation refined to a limit, and the dx sits where the strip width δx used to. [2]