inkMaths

ANSWER BOOK

Centres of mass by integration

Further Mechanics 2 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 $x(G) = \int xy\,dx \div \int y\,dx$ and $y(G) = \int \tfrac{1}{2}y^2\,dx \div \int y\,dx$. A vertical strip of height y acts at horizontal distance x, giving the moment $xy\,\delta x$. But it acts at its own halfway height $y/2$, so the vertical moment is $y \times y/2\,\delta x$, hence the half and the square. [2]

A2 Area = $\int(4 - x^2)\,dx = 8 - 8/3 =$ 16/3. $\int xy\,dx = \int(4x - x^3)\,dx = 8 - 4 = 4$, so $x(G) = 4 \div 16/3 =$ 0.75. $\int \tfrac{1}{2}y^2\,dx = \tfrac{1}{2}(32 - 64/3 + 32/5) = 8.533$, so $y(G) = 8.533 \div 16/3 =$ 1.6. [3]

SECTION B · Standard

B1 By symmetry the centre of mass lies on the axis, so only $x(G)$ is needed. Volume $\propto \int y^2\,dx = \int x\,dx =$ 8. Moment $\propto \int xy^2\,dx = \int x^2\,dx =$ 64/3. So $x(G) = (64/3) \div 8 =$ $8/3 \approx 2.67$, two thirds of the way along, since the solid widens towards the far end. [4]

B2 Volume $\propto \int(r^2 - x^2)\,dx$ from 0 to $r = r^3 - r^3/3 =$ $2r^3/3$. Moment $\propto \int x(r^2 - x^2)\,dx = r^4/2 - r^4/4 =$ $r^4/4$. Dividing: $x(G) = (r^4/4) \div (2r^3/3) =$ $3r/8$. It lies well inside the flat face, because the wide end is there. [4]

B3 Each element carries a mass $\rho\,dV$ rather than dV, so the density goes inside both integrals: the total mass is $\int \rho\,dV$ and the moment is $\int x\rho\,dV$. Everything else is unchanged, since the centre of mass is still the mass-weighted average of position. A constant ρ cancels, which is why uniform bodies never need it. [3]

SECTION C · Stretch

C1 Mass = $\int(1 + x)\,dx$ from 0 to $2 = 2 + 2 =$ 4 kg. Moment = $\int x(1 + x)\,dx = 2 + 8/3 =$ 14/3. So $x(G) = (14/3) \div 4 =$ $7/6 \approx 1.17$ m. It sits past the midpoint, towards the denser end, which is the check to make. [4]