inkMaths

ANSWER BOOK

De Moivre's theorem and trigonometric identities

Complex numbers · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$; for a general number, raise the modulus to the power n and multiply the argument by n. The theorem speaks about moduli and arguments, so a number given as $x + yi$ must be converted first. [2]

A2 $1 - i$ has modulus $\sqrt{2}$ and argument $-\pi/4$. The sixth power has modulus $(\sqrt{2})^6 = 8$ and argument $-3\pi/2$, which reduces to $\pi/2$. So $(1 - i)^6 = 8i$. Bracket-expansion agrees: $(1 - i)^2 = -2i$, and $(-2i)^3 = 8i$. [3]

SECTION B · Standard

B1 $\sqrt{3} + i$ has modulus 2 and argument $\pi/6$. The fifth power has modulus $2^5 = 32$ and argument $5\pi/6$. Converting back: $32(\cos 5\pi/6 + i\sin 5\pi/6) = 32(-\sqrt{3}/2 + i/2) = -16\sqrt{3} + 16i$. [4]

B2 By De Moivre, the power multiplies the argument: $\cos 6\pi/12 + i\sin 6\pi/12 = \cos\pi/2 + i\sin\pi/2 = i$. Unit modulus stays unit; only the angle moves. [3]

B3 The binomial expansion gives $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$, and De Moivre says the total is $\cos 3\theta + i\sin 3\theta$. Imaginary parts: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$. Replace $\cos^2\theta$ with $1 - \sin^2\theta$: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [4]

SECTION C · Stretch

C1 Expand $(\cos\theta + i\sin\theta)^4$ and take real parts: $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$. Substitute $\sin^2\theta = 1 - \cos^2\theta$: $\cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2 = 8\cos^4\theta - 8\cos^2\theta + 1$. A numerical spot check at $\theta = 45°$: both sides give -1. [4]