inkMaths

ANSWER BOOK

Diagonalisation and the Cayley-Hamilton theorem

Further Pure 2 · Year FM

6 questions · 21 marks

NAVY EDITION

SECTION A · Warm-up

A1 Put the eigenvectors in as columns: P has rows $(1, 1)$ and $(1, -2)$. Then D is diagonal with the matching eigenvalues in the same order, 5 then 2. Swapping the columns of P works provided the entries of D are swapped with them; mismatching the order is the standard error. [3]

A2 $M^5 = PD^5P^{-1}$ with D^5 diagonal, entries 3125 and 32. Since $\det P = -3$, P^{-1} is $(-1/3)$ times the matrix with rows $(-2, -1)$ and $(-1, 1)$. Multiplying out gives M^5 with rows (2094, 1031) and (2062, 1063). Only the diagonal entries were ever raised to a power. [3]

SECTION B · Standard

B1 Every square matrix satisfies its own characteristic equation. Here that equation is $\lambda^2 - 7\lambda + 10 = 0$, so the claim is $M^2 - 7M + 10I = 0$. M^2 has rows $(18, 7)$ and $(14, 11)$; $7M$ has rows $(28, 7)$ and $(14, 21)$; and $10I$ has 10 on the diagonal. Adding gives the zero matrix, as required. [4]

B2 From $M^2 - 7M + 10I = 0$, multiply throughout by M^{-1}: $M - 7I + 10M^{-1} = 0$, so $M^{-1} = (7I - M)/10$. That is one tenth of the matrix with rows $(3, -1)$ and $(-2, 4)$. Checking, M times that matrix is $10I$, so dividing by 10 gives the identity. The method needs no cofactors. [4]

B3 Take the matrix with rows $(1, 1)$ and $(0, 1)$. Its characteristic equation is $(1 - \lambda)^2 = 0$, so 1 is a repeated eigenvalue, but solving $(M - I)v = 0$ gives only multiples of $(1, 0)$. With one independent eigenvector there is no invertible P of eigenvectors, so no diagonalisation exists. A repeated eigenvalue is a warning, not a verdict: some repeated cases still supply enough eigenvectors. [3]

SECTION C · Stretch

C1 From $M^2 = 7M - 10I$: $M^3 = 7M^2 - 10M = 7(7M - 10I) - 10M = 39M - 70I$. Then $M^4 = 39M^2 - 70M = 39(7M - 10I) - 70M =$ 203M - 390I, so $a = 203$ and $b = -390$. Checking, that gives rows $(422, 203)$ and $(406, 219)$, which agrees with squaring M^2 directly. Every power reduces to a linear combination of M and I. [4]