inkMaths

ANSWER BOOK

Eigenvalues and eigenvectors

Further Pure 2 · Year FM

6 questions · 21 marks

NAVY EDITION

SECTION A · Warm-up

A1 The characteristic equation is $(4 - \lambda)(3 - \lambda) - 2 = 0$, that is $\lambda^2 - 7\lambda + 10 = 0$, so $\lambda = 5$ or $\lambda = 2$. For $\lambda = 5$ the first row gives $-x + y = 0$, so $(1, 1)$ works. For $\lambda = 2$ it gives $2x + y = 0$, so $(1, -2)$ works. Any non-zero multiple of either is equally valid. [3]

A2 Start from $Mv = \lambda v$. Then $M^2v = M(Mv) = M(\lambda v) = \lambda(Mv) = \lambda(\lambda v) = \lambda^2 v$. Since v is non-zero, λ^2 is an eigenvalue of M^2 and v is an eigenvector for it. The same argument repeated gives λ^k for M^k. [3]

SECTION B · Standard

B1 Expanding along the middle row, the characteristic equation factorises as $(3 - \lambda)[(2 - \lambda)^2 - 1] = 0$. The bracket gives $2 - \lambda = \pm 1$, so $\lambda = 1$ or $\lambda = 3$. The eigenvalues are 1 and 3, with 3 repeated. For $\lambda = 1$, $(1, 0, -1)$ works; for $\lambda = 3$, both $(1, 0, 1)$ and $(0, 1, 0)$ do, so that eigenvalue has two independent eigenvectors. [4]

B2 The sum of the eigenvalues equals the trace, and their product equals the determinant: both follow from comparing $\lambda^2 - (\text{trace})\lambda + (\det)$ with the factorised form. For rows $(4, 1)$ and $(2, 3)$ the trace is 7 and $5 + 2 = 7$; the determinant is $12 - 2 = 10$ and $5 \times 2 = 10$. It is the fastest check on a pair of eigenvalues you have just found. [4]

B3 Each eigenvector spans a line through the origin that the transformation maps onto itself: an invariant line. Points on it are stretched by the factor λ without changing direction, and a negative λ reverses them along the same line. A matrix with no real eigenvalues, such as a rotation through 90°, has no invariant line at all. [3]

SECTION C · Stretch

C1 $\lambda^2 - 6\lambda + 8 = 0$ gives $\lambda = 4$ with eigenvector $(1, 1)$ and $\lambda = 2$ with eigenvector $(1, -1)$, whose scalar product is 0. In general, if $Su = \lambda u$ and $Sv = \mu v$ with $\lambda \neq \mu$, then $u \cdot (Sv) = (Su) \cdot v$ because S is symmetric, so $\mu(u \cdot v) = \lambda(u \cdot v)$. Since $\lambda \neq \mu$ this forces $u \cdot v = 0$. [4]