inkMaths

ANSWER BOOK

Graphs: order, Eulerian paths and planarity

Decision Mathematics 1 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 Every edge contributes one to the order of each of its two ends, so the orders of all the nodes total twice the number of edges, which is even. The even-order nodes contribute an even amount, so the odd-order ones must too, and that is only possible if there is an even number of them. [2]

A2 The orders total 18, which is twice the nine edges, as required. Four nodes are odd, and four is more than two, so the graph is neither Eulerian nor semi-Eulerian: no trail can use every edge exactly once. [3]

SECTION B · Standard

B1 K_n has n(n - 1)/2 edges, so K5 has 10 and K6 has 15. In K_n every node has order n - 1. For K5 that is 4, even, so K5 is Eulerian. For K6 it is 5, odd, and all six nodes are odd, so K6 is neither Eulerian nor semi-Eulerian. [4]

B2 It needs a Hamiltonian cycle, one visiting every node exactly once. Redraw that cycle as a circle with the remaining edges as chords, then take the chords in turn, placing each inside or outside the circle so that it crosses nothing already placed. If every chord can be placed the graph is planar and the drawing proves it. If some chord conflicts with what is inside and also with what is outside, no placement exists and the graph is not planar. [4]

B3 Isomorphism requires a one-to-one correspondence of nodes that preserves every edge, and matching totals do not supply one. The orders could be arranged differently: in one graph the two order-4 nodes might be joined and in the other not. To prove isomorphism you must give the correspondence explicitly and check every edge. [3]

SECTION C · Stretch

C1 With exactly two odd nodes the network is semi-Eulerian, so a trail using every path exactly once exists, but it must start at one odd junction and finish at the other: the warden cannot return to the start. Adding a path between those two junctions raises both orders by one, making every node even, so the network becomes Eulerian and a closed round exists. The warden then walks one extra path but ends where they began. [4]