inkMaths

ANSWER BOOK

Groups and their axioms

Further Pure 2 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 Closure: the operation on two elements of the set gives an element of the set. Associativity: $(ab)c = a(bc)$ for all elements. Identity: there is an element e with $ea = ae = a$ for every a. Inverses: every a has an element a^{-1} in the set with $aa^{-1} = a^{-1}a = e$. Commutativity is not required. [2]

A2 The products are $3 \times 5 = 15 \equiv 7$, $3 \times 7 = 21 \equiv 5$ and $5 \times 7 = 35 \equiv 3$, and each element squares to 1, so the set is closed. Multiplication modulo 8 is associative, 1 is the identity, and every element is its own inverse. The identity has order 1; 3, 5 and 7 each have order 2. [3]

SECTION B · Standard

B1 Rows for 1, 2, 3, 4 read (1, 2, 3, 4), (2, 4, 1, 3), (3, 1, 4, 2) and (4, 3, 2, 1). Every entry lies in the set, the identity 1 appears exactly once in each row and column so inverses exist, and modular multiplication is associative. Powers of 2 give 2, 4, 3, 1, so 2 has order 4 and generates the whole group: it is cyclic. (3 works as well; 4 has order 2 and does not.) [4]

B2 The order of a group is the number of elements it contains; the order of an element is the least positive power of it that gives the identity. The symmetry group of an equilateral triangle has order 6: three rotations and three reflections. The rotation through 120° has order 3, since three of them return the triangle to its starting position, while each reflection has order 2. [3]

B3 Multiplying the matrix for m by the matrix for n gives rows $(1, m + n)$ and $(0, 1)$, so the set is closed and the operation behaves like adding the integers. Matrix multiplication is associative, $n = 0$ gives the identity matrix, and the matrix for $-n$ inverts the matrix for n. Since the operation is addition of the n values, the group is isomorphic to the integers under addition. [4]

SECTION C · Stretch

C1 With 0 included there is no inverse for 0: no residue multiplied by 0 gives the identity 1. Removing it leaves {1, 2, 3}, but $2 \times 2 = 4 \equiv 0$, which is outside the set, so closure fails and 2 has no inverse either. The multiplicative residues modulo n form a group only when the residues kept are those coprime to n, which for 4 means {1, 3}. [4]