inkMaths

ANSWER BOOK

Induction: divisibility and matrices

Further proof · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 $f(1) = 8 - 1 = 7$, which is divisible by 7. Hypothesis: assume $f(k) = 8^k - 1$ is divisible by 7 for some positive integer k. [2]

A2 $f(k + 1) - 8f(k) = (8^{k+1} - 1) - (8^{k+1} - 8) = 7$. So $f(k + 1) = 8f(k) + 7$: a multiple of 7 plus 7, divisible by 7. With the base case, 7 divides $8^n - 1$ for all n. [3]

SECTION B · Standard

B1 Base: $3^2 + 7 = 16 = 8 \times 2$. Step: with $f(k) = 3^{2k} + 7$ assumed divisible by 8, $f(k + 1) = 9 \times 3^{2k} + 7 = 9f(k) - 56$. Both terms are multiples of 8 ($56 = 8 \times 7$), so $f(k + 1)$ is too. The multiplier 9 comes from $3^{2(k+1)} = 9 \times 3^{2k}$: each step multiplies the power part by 9. [4]

B2 Base: M^1 has rows (1, 1), (0, 1), matching n = 1. Step: assume M^k has rows (1, k), (0, 1). Then $M^{k+1} = M^kM$ multiplies to rows (1, k + 1), (0, 1): the top-right entry is $1 \times 1 + k \times 1 = k + 1$ and the rest are unchanged. So the pattern holds for all n: the shear just accumulates its slide. [4]

B3 Base: D^1 has rows (2, 0), (0, 3) = $(2^1, 0)$, $(0, 3^1)$. Step: assume the result at k; multiplying by D scales the top-left entry by 2 and the bottom-right by 3, giving 2^{k+1} and 3^{k+1} with the zeros preserved. Diagonal matrices power entry by entry, and induction is how that sentence is earned. [4]

SECTION C · Stretch

C1 Choose m to match the growth of the power term, so the powers cancel: for $8^n - 1$ take m = 8, for $3^{2n} + 7$ take m = 9. What remains is a constant, and if that constant is divisible by the target then $f(k + 1) = m\,f(k) + \text{constant}$ inherits divisibility from the hypothesis. [3]