inkMaths

ANSWER BOOK

Leibnitz's theorem and the Weierstrass substitution

Further Pure 1 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 The nth derivative of fg is the sum over r from 0 to n of ${}^nC_r$ times the rth derivative of f times the (n - r)th derivative of g. The coefficients are binomial because each differentiation chooses which factor to act on. [2]

A2 $\sin x = 2t/(1 + t^2)$, $\cos x = (1 - t^2)/(1 + t^2)$ and $dx = 2\,dt/(1 + t^2)$. The dx conversion is the one most often forgotten, and it changes every answer. [3]

SECTION B · Standard

B1 Take $f = x^3$, whose derivatives are $3x^2$, $6x$, 6 and then zero. Only four terms survive: $x^3 + {}^4C_1(3x^2) + {}^4C_2(6x) + {}^4C_3(6)$, all multiplied by e^x. That is $e^x(x^3 + 12x^2 + 36x + 24)$. Differentiating four times by hand gives the same polynomial with far more writing. [4]

B2 $1 + \cos x = 2/(1 + t^2)$, so the integrand times dx becomes $(1 + t^2)/2 \times 2\,dt/(1 + t^2) = dt$. The limits transform: $x = 0$ gives $t = 0$, and $x = \pi/2$ gives $t = 1$. The integral is therefore just [t] from 0 to 1 = 1. The substitution reduced the whole problem to integrating 1. [4]

B3 $\sec x\,dx = (1 + t^2)/(1 - t^2) \times 2\,dt/(1 + t^2) = 2\,dt/(1 - t^2)$. Partial fractions: $2/((1 - t)(1 + t)) = 1/(1 - t) + 1/(1 + t)$, integrating to $-\ln|1 - t| + \ln|1 + t| = \ln|(1 + t)/(1 - t)| + c$. Evaluated from 0 to $\pi/3$, with $t = 1/\sqrt{3}$, it gives 1.3170. [4]

SECTION C · Stretch

C1 A polynomial of degree k has zero as its (k + 1)th derivative and beyond, so every term of the Leibnitz sum involving those derivatives vanishes. However large n is, only k + 1 terms survive, which turns an n-fold differentiation into a short finite sum. [3]