



ANSWER BOOK

Locating roots and iteration

Numerical methods · Year 13

8 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

- A1** $f(1) = 1 + 1 - 5 = -3 < 0$ and $f(2) = 8 + 2 - 5 = 5 > 0$. There is a change of sign, and f is continuous, so a root lies in $(1, 2)$. Both the sign change and the continuity must be stated.
- A2** The function might be discontinuous: $1/x$ changes sign between $x = -1$ and $x = 1$ with no root, because of the asymptote at 0. The argument needs f to be continuous on the interval.
- A3** From $x^3 + x - 5 = 0$, isolate the cube: $x^3 = 5 - x$, then take the cube root of both sides: $x = \sqrt[3]{5 - x}$. Any correct isolation earns the marks; this one converges.

SECTION B · Standard

- B1** $x_1 = (5 - 1.5)^{1/3} = 3.5^{1/3} = 1.51829$. $x_2 = (5 - 1.51829)^{1/3} = 1.51564$. $x_3 = (5 - 1.51564)^{1/3} = 1.51603$. The iterates close in on each other, alternating sides of the root.
- B2** Test the interval that rounds to 1.516: $f(1.5155) = -0.0038... < 0$ and $f(1.5165) = 0.0041... > 0$. The sign change with continuity traps the root in $(1.5155, 1.5165)$, and every number there rounds to 1.516. Checking both ends of the rounding interval is everything the method asks.
- B3** The gradient of g is small there: $|g'(\alpha)| \approx 0.145$, and an iteration $x = g(x)$ converges near a root when $|g'| < 1$. The negative sign of g' is what makes the path spiral from alternate sides rather than climb a staircase.

SECTION C · Stretch

- C1** $x_1 = 5 - 1.5^3 = 5 - 3.375 = 1.625$, then $x_2 = 5 - 1.625^3 = 0.708...$, and the values swing ever further from the root. Here $g(x) = 5 - x^3$ has $g'(x) = -3x^2$, about -6.9 near the root, and $|g'| > 1$ throws the iterates away: a valid rearrangement is not always a useful one.
- C2** If $x_n \rightarrow \alpha$ then, since g is continuous, taking limits in $x_{n+1} = g(x_n)$ gives $\alpha = g(\alpha)$, which rearranges back to $f(\alpha) = 0$. But settling digits show only that successive iterates agree with each other; the accuracy claim needs the sign-change check on the rounding interval of the claimed value.