inkMaths

ANSWER BOOK

Mean values and improper integrals

Further calculus · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 The integral of f from a to b divided by b - a. It is the height of the rectangle on the same base with the same area as the region under the curve: the level the curve would settle to if its area were poured flat. [2]

A2 $\int \sin x\, dx$ from 0 to π = [-cos x] = 2. Dividing by the width π gives mean $2/\pi \approx 0.637$. Less than the maximum 1, as the curve spends its time rising to the peak and falling away. [3]

SECTION B · Standard

B1 $\int e^{2x}\, dx$ from 0 to 1 = $[e^{2x}/2] = (e^2 - 1)/2$. The width is 1, so the mean is $(e^2 - 1)/2 \approx 3.19$. The mean sits well below the endpoint value $e^2 \approx 7.39$ because an exponential spends most of any interval far below its final value. [4]

B2 Integrate to a limit t: $[-2x^{-1/2}]$ from 1 to t = $2 - 2/\sqrt{t}$. As $t \to \infty$ the second term vanishes, so the integral converges to 2. The power 3/2 is above the boundary case 1, which is what buys convergence. [4]

B3 The integrand is unbounded at 0, so integrate from t to 4: $[2\sqrt{x}] = 4 - 2\sqrt{t}$. As $t \to 0^+$ this tends to 4, so the integral converges to 4. The spike at zero is infinitely tall but narrow enough to hold finite area. [4]

SECTION C · Stretch

C1 The $1/x^{3/2}$ integral converges (to 2); the 1/x integral gives ln t, which grows without bound. Tending to zero is not enough: what matters is how fast. Powers of x below -1 shrink fast enough to trap finite area, and 1/x sits exactly on the divergent side of the boundary. [3]