inkMaths

ANSWER BOOK

Probability generating functions

Further Statistics 1 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 $G(t) = E(t^X) = \Sigma t^x P(X = x)$. Setting $t = 1$ makes every power 1, so $G(1)$ is the total probability, which is always 1. It is the quickest check on any generating function you write down. [2]

A2 This is the binomial form $(q + pt)^n$, so $X \sim B(8, 0.4)$. $G'(t) = 8(0.4)(0.6 + 0.4t)^7$, and at $t = 1$ the bracket is 1, so $E(X) = 3.2$. That agrees with $np = 8 \times 0.4$. [3]

SECTION B · Standard

B1 $G''(t) = 8(7)(0.4)^2(0.6 + 0.4t)^6$, so $G''(1) = 56 \times 0.16 = 8.96$. Then $Var(X) = G''(1) + G'(1) - [G'(1)]^2 = 8.96 + 3.2 - 10.24 = 1.92$, which matches $npq = 8(0.4)(0.6)$. [4]

B2 $G(t) = \Sigma t^x e^{-\lambda}\lambda^x/x! = e^{-\lambda}\Sigma(\lambda t)^x/x! = e^{-\lambda}e^{\lambda t} = e^{\wedge}(\lambda(t - 1))$. Differentiating: $G'(t) = \lambda e^{\wedge}(\lambda(t - 1))$, so $G'(1) = \lambda$. Checking $G(1) = e^0 = 1$ confirms the derivation. [4]

B3 $G_{X+Y}(t) = e^{\wedge}(2(t - 1)) \times e^{\wedge}(7(t - 1)) = e^{\wedge}(9(t - 1))$, which is the generating function of Po(9). Since generating functions determine distributions uniquely, $X + Y \sim Po(9)$. [3]

SECTION C · Stretch

C1 $G'(t) = p/(1 - qt)^2$, so $G'(1) = p/p^2 = 1/p$. Differentiating again: $G''(t) = 2pq/(1 - qt)^3$, so $G''(1) = 2pq/p^3 = 2q/p^2$. Then $Var = 2q/p^2 + 1/p - 1/p^2 = (2q + p - 1)/p^2$. Since $p - 1 = -q$, the numerator is $2q - q = q$, giving q/p^2 as required. With $p = 0.25$ that is 12, and the mean is 4. [4]