inkMaths

ANSWER BOOK

Reduction formulae

Further Pure 2 · Year FM

6 questions · 21 marks

NAVY EDITION

SECTION A · Warm-up

A1 Write the integrand as $\sin^{n-1}x \times \sin x$ and integrate by parts, differentiating the first factor: the boundary term is $[-\sin^{n-1}x \cos x]$, which vanishes at both limits. What is left is $(n - 1)\int\sin^{n-2}x \cos^2x\,dx$. Replacing $\cos^2x$ by $1 - \sin^2x$ splits it into $(n - 1)I_{n-2} - (n - 1)I_n$, so $I_n + (n - 1)I_n = (n - 1)I_{n-2}$, which is the result. [3]

A2 $I_4 = (3/4)I_2 = (3/4)(1/2)(\pi/2) = 3\pi/16$, about 0.5890. $I_5 = (4/5)I_3 = (4/5)(2/3)(1) = 8/15$, about 0.5333. Even n runs down to I_0 and keeps a π; odd n runs down to I_1 and does not. [3]

SECTION B · Standard

B1 Integrating by parts with $u = x^n$ gives $[x^ne^x] - n\int x^{n-1}e^x\,dx$. The boundary term is e at x = 1 and 0 at x = 0, so $I_n = e - nI_{n-1}$. Starting from $I_0 = e - 1$: $I_1 = e - (e - 1) = 1$, $I_2 = e - 2$, and $I_3 = e - 3(e - 2) = 6 - 2e$, about 0.5634. [4]

B2 Split off $\tan^2x = \sec^2x - 1$, giving $\int\tan^{n-2}x \sec^2x\,dx - I_{n-2}$. The first integral is $[\tan^{n-1}x/(n - 1)]$, which is $1/(n - 1)$ at $\pi/4$ and 0 at 0. So $I_n = 1/(n - 1) - I_{n-2}$. With $I_0 = \pi/4$: $I_2 = 1 - \pi/4$, and $I_4 = 1/3 - (1 - \pi/4) = \pi/4 - 2/3$, about 0.1187. [4]

B3 The formula only relates one integral to another, so it generates a chain rather than a number. You need one integral evaluated directly to anchor it. The index drops by a fixed step, so follow the chain down: a step of 2 lands on I_0 for even n and I_1 for odd n, and a step of 1 always lands on I_0. [3]

SECTION C · Stretch

C1 By parts with $u = x^n$: the boundary term $[-x^ne^{-x}]$ is 0 at the origin and tends to 0 at infinity, since the exponential beats any power. What remains is $n\int x^{n-1}e^{-x}\,dx = nI_{n-1}$. Since $I_0 = 1$, repeated use gives $I_n = n!$, so $I_5 = 120$. [4]