inkMaths

ANSWER BOOK

Roots of unity and complex roots

Complex numbers · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 Exactly five, the fifth roots of 32. Each has modulus $32^{(1/5)} = 2$; they differ only in argument, spaced $2\pi/5$ apart round the circle of radius 2. [2]

A2 Arguments $0, \pi/3, 2\pi/3, \pi, -2\pi/3, -\pi/3$: a regular hexagon on the unit circle with one vertex at 1. Their sum is 0, by the rotational symmetry of the set. [3]

SECTION B · Standard

B1 -27 has modulus 27 and argument π, so one root has modulus 3 and argument $\pi/3$: $z = 3(\cos \pi/3 + i \sin \pi/3) = 3/2 + (3\sqrt{3}/2)i$. The others sit $2\pi/3$ round: $z = -3$ (argument π) and the conjugate $3/2 - (3\sqrt{3}/2)i$. Three roots, an equilateral triangle with the real root at -3. [4]

B2 Modulus 16 and argument π, so each root has modulus 2 with arguments $\pi/4, 3\pi/4, -3\pi/4, -\pi/4$: $z = \pm\sqrt{2} \pm \sqrt{2}\,i$, all four sign combinations. They form a square of circumradius 2 with vertices on neither axis, one in each quadrant. [4]

B3 $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$. Any root other than 1 kills the second bracket, so the four non-trivial roots satisfy $z^4 + z^3 + z^2 + z + 1 = 0$; adding 1's contribution, the full sum of all five roots is the negated z^4-coefficient of the quintic, which is 0. [3]

SECTION C · Stretch

C1 $64i$ has modulus 64 and argument $\pi/2$, so one root has modulus 4 and argument $\pi/6$: $z = 4(\cos \pi/6 + i \sin \pi/6) = 2\sqrt{3} + 2i$. Spacing the rest $2\pi/3$ apart gives arguments $5\pi/6$ and $-\pi/2$: $z = -2\sqrt{3} + 2i$ and $z = -4i$. Check the last: $(-4i)^3 = -64i^3 = 64i$. [4]