inkMaths

ANSWER BOOK

Series solutions of differential equations

Further Pure 1 · Year FM

6 questions · 20 marks

NAVY EDITION

SECTION A · Warm-up

A1 Rearrange for the highest derivative and substitute the initial values to find it at the starting point. Differentiate the whole equation and substitute again for the next derivative, and so on. Each derivative becomes one coefficient of the Taylor series of the solution. [2]

A2 dy/dx at 0 is $0 + 1^2 = 1$. Differentiating the equation: the second derivative is $1 + 2y(dy/dx)$, which at 0 is $1 + 2(1)(1) = 3$. [3]

SECTION B · Standard

B1 Differentiating $1 + 2y\,y'$ gives $2(y')^2 + 2y\,y''$, which at 0 is $2(1) + 2(1)(3) = 8$. So $y = 1 + x + 3x^2/2! + 8x^3/3!$, that is $y = 1 + x + 1.5x^2 + (4/3)x^3 + \dots$. Keeping the factorials visible until the last line makes the method obvious to a marker. [4]

B2 $dz/dx = 2y(dy/dx)$, so the equation becomes $dz/dx + z = x$: linear in z. The integrating factor is e^x, giving $e^x z = \int x e^x\,dx = (x - 1)e^x + C$. So $z = x - 1 + Ce^{-x}$, and $y^2 = x - 1 + Ce^{-x}$. Substituting back into the original equation confirms it. [4]

B3 The equation delivers the second derivative in terms of x, y and dy/dx, so both y and dy/dx must be known at the starting point before anything can be evaluated. With only one condition the very first substitution is impossible. [3]

SECTION C · Stretch

C1 At 0: $y'' = -0 - 2 = -2$. Differentiating: $y''' = -y' - xy'' - 2y' = -3y' - xy''$, which is 0 at the origin. Once more: the fourth derivative is $-4y'' - xy'''$, which at 0 is 8. So $y = 1 - 2x^2/2! + 8x^4/4! = 1 - x^2 + x^4/3 + \dots$. The odd terms all vanish because $y'(0) = 0$ and the equation preserves that symmetry. [4]