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ANSWER BOOK

Subgroups, Lagrange's theorem and isomorphism

Further Pure 2 · Year FM

6 questions · 20 marks

NAVY EDITION

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SECTION A · Warm-up

A1 The order of any subgroup of a finite group divides the order of the group. Since the powers of a single element form a subgroup, the order of every element divides the order of the group as well. [2]

A2 The subgroups are $\{e\}$, $\{e, g^3\}$, $\{e, g^2, g^4\}$ and the whole group, of orders 1, 2, 3 and 6. Each of those divides 6, as Lagrange requires, and every divisor of 6 is achieved: that always happens in a cyclic group, though not in general. [3]

SECTION B · Standard

B1 Subgroup orders must divide 8, so they are 1, 2, 4 or 8. If some element a had order 3 then $\{e, a, a^2\}$ would be a subgroup of order 3, and 3 does not divide 8. That contradicts Lagrange, so no such element exists. [4]

B2 By Lagrange any subgroup has order dividing p, so its order is 1 or p: the only subgroups are the identity and the whole group. Take any element a other than the identity. The subgroup generated by a is not $\{e\}$, so it must be the whole group, and therefore a generates G. Hence G is cyclic, and every non-identity element is a generator. [4]

B3 They are not. The cyclic group of order 4 contains an element of order 4, its generator. In $\{1, 3, 5, 7\}$ every element squares to 1, so no element has order more than 2. An isomorphism preserves the order of every element, so no bijection between them can be one. [3]

SECTION C · Stretch

C1 The four rotations, through 0°, 90°, 180° and 270°, form a cyclic subgroup of order 4. The identity, the rotation through 180° and the two reflections in the lines through opposite edge midpoints form a different subgroup of order 4, in which every element squares to the identity, so the two are not isomorphic. The identity with the rotation through 180° gives a subgroup of order 2. All three orders divide 8. [4]