inkMaths

ANSWER BOOK

The structure of proof: deduction and exhaustion

Proof · Year 12–13

8 questions · 22 marks

NAVY EDITION

SECTION A · Warm-up

A1 [2] An even number is $2n$ and an odd number is $2n + 1$, where n is an integer. Consecutive odd numbers are $2n + 1$ and $2n + 3$. Declaring that n is an integer is part of the answer, because the letter is there precisely to stand for every integer at once.

A2 [2] Fifty checks are evidence, and evidence is not proof: case 51 could still fail. A proof must be a general argument that covers every case at once, by deduction from known facts, or by exhaustion when the cases form a complete finite list.

SECTION B · Standard

B1 [3] Complete the square: $n^2 + 4n + 7 = (n + 2)^2 + 3$. A square is never negative, so $(n + 2)^2 \ge 0$, and therefore $(n + 2)^2 + 3 \ge 3$ > 0. Rewriting the expression so that its sign becomes visible is what the completing-the-square step is for.

B2 [3] Let the numbers be $2n + 1$ and $2n + 3$ for an integer n. Their sum is $4n + 4 = 4(n + 1)$, and $n + 1$ is an integer, so the sum is 4 times an integer. The final sentence matters: exhibiting the factor of 4 and saying the other factor is an integer is what “divisible by 4” means.

B3 [3] $(2n + 3)^2 - (2n + 1)^2 = (4n^2 + 12n + 9) - (4n^2 + 4n + 1) = 8n + 8 = 8(n + 1)$, which is 8 times an integer. Expanding both squares fully before subtracting keeps the sign errors out.

B4 [3] Every integer is even or odd, so two cases cover everything. If $k = 2n$ then $k^2 = 4n^2$, a multiple of 4. If $k = 2n + 1$ then $k^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1$, one more than a multiple of 4. This is proof by exhaustion over types rather than individual numbers: the two cases are exhaustive because every integer is one or the other.

SECTION C · Stretch

C1 [4] The complete list is 7, 11, 13, 17, 19. In turn, $p^2 - 1$ gives 48, 120, 168, 288, 360, and dividing by 24 gives 2, 5, 7, 12, 15, all integers. Every case on the complete list has been checked, so the claim is proved. Listing the primes correctly is the first mark: miss one and the proof collapses.

C2 [2] Exhaustion needs a complete finite list of cases, and the integers go on for ever: however many are checked, infinitely many remain. A claim about all integers needs a general argument, which is why the deduction proof via the completed square is the only route.